Optimized Product Quantization

Tiezheng Ge, Kaiming He, Qifa Ke, and Jian Sun

MSRA
Introduction

- Compact Coding for ANN Search
  - Memory
    - 128-d float: 512 bytes → 16 bytes
    - 1 billion items: 512 GB → 16 GB
  - Time
    - Computation: x10-x100 faster
    - Transmission (disk/web): x30 faster
Background

- Vector Quantization (VQ)

\[ x \rightarrow c_i \rightarrow i(x) \]

- nearest codeword
- code stored

Number of codewords: \( K \)

Code length: \( B = \log_2 K \)
Background

• VQ for ANN Search

\[ d(x, y) \approx d(c_i, c_j) \triangleq lut(i, j) \]
Background

• K-means

$$\min_c \sum_x \|x - c_i(x)\|^2$$ (distortion)

😊

• Minimal distortion

😊

• Intractable look-up: $K = 2^B$
Background

- **Product Quantization (PQ)** [PAMI 2011]

\[
\min_{C_1, \ldots, C_M} \sum_x \| x - c_i(x) \|^2
\]

s.t. \( c \in C_1 \times C_2 \times \cdots \times C_M \)

- Huge codebook: \( K = k^M \)
- Tractable: \( M \) \( k \)-by-\( k \) tables
- Sensitive to projection
Background

• Iterative Quantization (ITQ) [CVPR 2011]

\[
\min_R \sum_x \| x - c_i(x) \|^2
\]

s.t. \( Rc \in \{-1,1\}^D, R^T R = I \)

😊

• Optimized wrt \( R \)

😊

• 1-d subspace

😊

• \( k = 2 \) only

😊

• no look-up
Our method

• Optimized Product Quantization (OPQ) [CVPR 2013]

\[
\min_{R,C_1,\ldots,C_M} \sum_{x} \|x - c_i(x)\|^2
\]

s.t. \( Rc \in C^1 \times C^2 \ldots \times C^M, R^T R = I \)

😊
• Huge codebook: \( K = k^M \)
• Tractable: \( M \) \( k \)-by-\( k \) tables
• High-dim subspace
• \( k \geq 2 \)
• Optimize wrt \( R \)
Relations

\[
\min_{C^1, \ldots, C^M} \sum_{x} \|x - c(x)\|^2
\]

s.t. \( c \in C^1 \times C^2 \ldots \times C^M \)

OPQ vs. PO

\[
\min_{R} \sum_{x} \|x - c(x)\|^2
\]

s.t. \( Rc \in \{-1,1\}^D \)

OPQ vs. ITQ

\[
\min_{R, C^1, \ldots, C^M} \sum_{x} \|x - c(x)\|^2
\]

s.t. \( Rc \in C^1 \times C^2 \ldots \times C^M \)

Challenges coupled \( R \) and \( C^1, \ldots, C^M \)
Solutions

• Challenges - coupled $R$ and $C^1, \ldots, C^M$
• Solution I
  – decoupling
  – fix $R$ solve $C^1, \ldots, C^M$
  – fix $C^1, \ldots, C^M$ solve $R$
• Solution II
  – lower bound: involves $R$ only
  – minimize lower bound w.r.t $R$
Solution I

• Step 1: fix $R$, solve for $C^1, ..., C^M$

$$
\min_{C^1, \ldots, C^M} \sum_x \| \hat{x} - \hat{c}_{i(x)} \|^2
$$

s.t. \( \hat{c} \in C^1 \times C^2 \ldots \times C^M \)

with $\hat{x} = Rx$, and $\hat{c} = Rc$

Standard PQ
in a projected space
Solution I

• Step 2: fix $C^1, \ldots, C^M$, solve for $R$

$$\min_R \sum_x \|Rx - \hat{c}_i(x)\|^2$$

s.t. $\hat{c} \in C^1 \times C^2 \ldots \times C^M$, $R^TR = I$

Rotate codewords without changing their relative positions
Solution I

• Step 2: fix $C^1, \ldots, C^M$, solve for $R$

$$\min_R \|RX - Y\|_F^2$$

s.t. $R^TR = I$

- Let $X = \{x\}, Y = \{y\}, y = \hat{c}_i(x)$
- $R = VU^T$, $[U, V] = \text{svd}(XY^T)$
Solution I

• Initialize
• Repeat
  – Fix $R$, solve:
    $$\min_{C_1, \ldots, C_M} \sum_x \|\hat{x} - \hat{c}_i(x)\|^2$$  (classical PQ)
  – Fix $C_1, \ldots, C_M$, solve:
    $$\min_R \|RX - Y\|_F^2$$  (classical ITQ)
• Until convergence
Solution II

• Decoupling
  – lower bound: involves $R$ only
  – minimize lower bound w.r.t $R$

• Assumes Gaussian distribution
  – analytical forms
  – theoretical guarantees
  – simple, non-iterative
Solution II

• Assumes $x \sim N(0, \Sigma)$
• $\hat{x} = Rx \sim N(0, \hat{\Sigma})$, with $\hat{\Sigma} = R\Sigma R^T$
• Decompose $\hat{x} = (\hat{x}^1, \hat{x}^2, ..., \hat{x}^M)$ into $M$ subspaces

$$\hat{x}^m \sim N(0, \hat{\Sigma}_{mm})$$

$$\hat{\Sigma} = \begin{pmatrix}
\hat{\Sigma}_{11} & \cdots & \hat{\Sigma}_{1M} \\
\vdots & \ddots & \vdots \\
\hat{\Sigma}_{M1} & \cdots & \hat{\Sigma}_{MM}
\end{pmatrix}$$
Solution II

• Rate distortion theory for $\hat{x}^m \sim N(0, \hat{\Sigma}_{mm})$

\[ E^m \geq k^{-2} \frac{\frac{M}{D} D}{M} |\hat{\Sigma}_{mm}|^{\frac{M}{D}} \]

  – Nearly achieved: k-means

• PQ distortion for $\hat{x} \sim N(0, \hat{\Sigma})$

\[ E \geq k^{-2} \frac{\frac{M}{D} D}{M} \sum_{m=1}^{M} |\hat{\Sigma}_{mm}|^{\frac{M}{D}} \]

• Optimize lower bound

\[ \min_{R} \sum_{m=1}^{M} |\hat{\Sigma}_{mm}|^{\frac{M}{D}} \]

s.t. $R^T R = I$
Solution II

• Optimization - achieve the lower bound of the lower bound

\[
\min_R \sum_{m=1}^M |\hat{\Sigma}_{mm}| \geq \sum |\hat{\Sigma}_{mm}| \geq M \prod |\hat{\Sigma}_{mm}| \geq M |\hat{\Sigma}| \equiv M |\Sigma|^{-1}
\]

AM-GM inequality     Fischer inequality
Solution II

- Optimization - achieve the lower bound of the lower bound

\[
\min_R \sum_{m=1}^M |\hat{\Sigma}_{mm}|^{\frac{M}{D}} \quad \text{s.t.} \quad R^T R = I
\]

\[
\sum |\hat{\Sigma}_{mm}|^{\frac{M}{D}} \geq M \prod |\hat{\Sigma}_{mm}|^{\frac{1}{D}} = M |\hat{\Sigma}|^{\frac{1}{D}} \equiv M |\Sigma|^{\frac{1}{D}}
\]

\[
\begin{array}{ccccc}
+ & + & \geq & + & \times \\
\times & \times & = & \times & \times \\
\times & \times & = & \times & \times \\
\end{array}
\]

Independent
Solution II

- Optimization - achieve the lower bound of the lower bound

\[
\min_R \sum_{m=1}^M |\hat{\Sigma}_{mm}|^{M/D} \quad \text{s.t.} \quad R^T R = I
\]

\[
\sum |\hat{\Sigma}_{mm}|^{M/D} = M \prod |\hat{\Sigma}_{mm}|^{1/D} = M |\hat{\Sigma}|^{1/D} \equiv M |\Sigma|^{1/D}
\]
Solution II

- Algorithm
  - *independent*: PCA
  - *balanced*:
    \[
    |\hat{\Sigma}_{11}| = \cdots = |\hat{\Sigma}_{mm}| = \cdots = |\hat{\Sigma}_{MM}|
    \]
  - \( |\hat{\Sigma}_{mm}| = \prod \{ \text{eigenvalues of } \hat{\Sigma}_{mm} \} \)
  - Greedy allocation:
    - Sort the eigenvalues of \( \Sigma \)
    - Prepare \( M \) buckets
    - Allocate the largest eigenvalue to the bucket having smallest product
Verification

- 64-d Gaussian, descending eigenvalues, 4 subspaces

\[ \hat{\Sigma} \]

\[ R \]

- not rotated
- random order
- random rotation
- forced balance
- solution I
- solution II

independent

- Yes
- Yes
- No
- No
- Yes
- Yes

balanced

- No
- almost
- almost
- Yes
- Yes
- Yes

distortion

- Yes
- Yes
- Yes
- Yes
Solution I vs. II

• I – non-parametric
  – Better fits non-Gaussian
  – Iterative (offline)
  – Needs init (e.g. by II)

• II – parametric
  – Guarantees for Gaussian
  – Solid theories
  – Non-iterative
  – Less well for non-Gaussian

• Best practice – solution I + solution II (initialize)
Experiments

• 1 million GIST, 100 NNs, exhaustive ranking
Experiments

• 1 million SIFT, 100 NNs, exhaustive ranking
Experiments

- 1 billion SIFT, 1 NNs, inverted indexing + re-ranking
  - Build inverted indexing via PQ  [Babenko, CVPR 2012]
  - Re-rank short lists via PQ  [Jegou, PAMI 2011]
  - We optimize both

<table>
<thead>
<tr>
<th></th>
<th>short list length</th>
<th>Recall@100</th>
<th>time</th>
</tr>
</thead>
<tbody>
<tr>
<td>[CVPR 2012]</td>
<td>10,000</td>
<td>74.8</td>
<td>7ms</td>
</tr>
<tr>
<td>Ours</td>
<td>10,000</td>
<td><strong>79.4</strong></td>
<td>7ms</td>
</tr>
<tr>
<td>[CVPR 2012]</td>
<td>100,000</td>
<td>96.0</td>
<td>49ms</td>
</tr>
<tr>
<td>Ours</td>
<td>100,000</td>
<td><strong>97.3</strong></td>
<td>49ms</td>
</tr>
</tbody>
</table>
Experiments

- **Image retrieval**
  - feature: VLAD [Jegou, PAMI 2011]
  - dataset: Holiday [Jegou, PAMI 2011]
  - ground truth: semantic

<table>
<thead>
<tr>
<th>memory / image</th>
<th>8 bytes</th>
<th>16 bytes</th>
<th>32 bytes</th>
</tr>
</thead>
<tbody>
<tr>
<td>mAP ($PQ_{RR}$)</td>
<td>38.1</td>
<td>47.9</td>
<td>53.0</td>
</tr>
<tr>
<td>mAP (OPQ)</td>
<td>47.7</td>
<td>52.2</td>
<td>54.3</td>
</tr>
</tbody>
</table>
Conclusion

- Excellent performance for ANN
- Solid theories
- Widely applicable